q-analogues of some prototype Berry phase calculations

1990 J. Phys. A: Math. Gen. 23 L951

(http://iopscience.iop.org/0305-4470/23/18/003)

J. Phys. A: Math. Gen. **23** (1990) L951-L955. Printed in the UK

LETTER TO THE EDITOR

q-analogues of some prototype Berry phase calculations

S K Soni

Physics Department, SGTB Khalsa College, University of Delhi, Delhi-110007, India

Received 4 July 1990

Abstract. Following the work of Macfarlane and Biedenharn we calculate Berry's phase for the $SU(2)_q$ model, the q-analogue of the displaced harmonic oscillator, and the generalized q-harmonic oscillator. We discuss the unusual features of our results.

Recently Biedenharn [1] and Macfarlane [2] have shown to the physics community an elegant and simple way of obtaining certain remarkable results in $SU(2)_q$, a quantum deformation of the universal enveloping algebra of the simplest Lie algebra SU(2). The really interesting issue is the physics behind the q structure. We take a small step in that direction in this letter. Using a q-analogue of the angular momentum and the boson operator calculus [1, 2], we shall q-extend certain known results for Berry phase calculations [3]. It may be recalled that Berry's phase refers to the extra phase that the wavefunction of a system can acquire under adiabatic cycling of the parameters upon which the Hamiltonian depends. This phase is geometrical in the sense that it depends only on the cycle the system follows in the space of its parameters, and not on the time taken for this adiabatic excursion.

Let us first consider spins in magnetic fields. A particle with spin j ($j = 0, \frac{1}{2}, 1, \ldots$) interacts with magnetic field via the Hamiltonian

$$H(\boldsymbol{B}) = k\hbar \boldsymbol{B} \cdot \boldsymbol{J} \tag{1}$$

where k is a constant involving the gyromagnetic ratio and $\boldsymbol{J}$ is the self-adjoint vector spin operator whose components J_x, J_y, J_z generate the commutator algebra of the 'quantum group' $SU(2)_q$ defined by the relations

$$[J_z, J_\pm] = \pm J_\pm \tag{2}$$

$$[J_+, J_-] = [2\, J_3]. \tag{3}$$

Here $J_+ = Jx \pm \mathrm{i} J_y$ and we have introduced the abbreviation

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sin h(sx)}{\sin h(s)} \tag{4}$$

where $q = \exp(s)$, s is real and positive, so that the right-hand side of (4) approaches x as $s \to 0$. This limit becomes the familiar classical limit $\hbar \to 0$ if we let $s = \hbar$. Thus we can say that (2) and (3) define a quantum deformation (using q as the deformation parameter) of the classical lie algebra of SU(2). Jimbo [4] has shown that there exists one representation of (2) and (3) for each spin j (integer or half-integer) with $2j+1$ eigenvalues m with integer spacing and that lie between $-j$ and $+j$. To return to our Berry's phase calculation, we consider the components of $\boldsymbol{B}$ as the external parameters and calculate the phase change $\gamma_m(C)$ of an eigenstate $|m, j(\boldsymbol{B})\rangle$ of the Hamiltonian (1) as $\boldsymbol{B}$ (and hence the spin pointing in that direction) is slowly rotated around a circuit C.

The quantum deformation has no effect on the energy eigenvalues

$$E_m(\boldsymbol{B}) = mk\hbar B. \tag{5}$$

Note that there is a $(2j+1)$-fold degeneracy when $\boldsymbol{B}=0$. The various matrix elements of $J_\pm$ in the $|jm\rangle$ basis are given by Jimbo. One has

$$J_\pm = \sum_{m=-j}^{+j} \sqrt{[j\mp m][j\pm m+1]}\,|j, m\pm 1\rangle\langle j, m|. \tag{6}$$

An interesting consequence of (6) can be read directly without any detailed calculation. The electronic Hamiltonian of a molecule in the neighbourhood of a diabolical point can be treated as a pseudospin-$\frac{1}{2}$ particle [3], for which the q-deformation does not modify the Hamiltonian and hence Berry's phase. Thus the geometrical phase factor is $(m=\pm\frac{1}{2})$

$$\exp\{\mathrm{i}\gamma_m(c)\} = \exp\{-\mathrm{i}m\Omega(C)\}, \tag{7}$$

where $\Omega(C)$ is the solid angle C subtended at $\boldsymbol{B}=0$, the location of the degeneracy (i.e. the diabolical point). In the case when $j>\frac{1}{2}$, the matrix elements of $J_\pm$ are deformed and the adiabatic phase factor becomes

$$\exp\{\mathrm{i}\gamma_m(C)\} = \exp\{-\tfrac{1}{2}[2m]\Omega(C)\}. \tag{8}$$

To see this, recall that Berry has shown that the points of degeneracy in the parameter space act as sources of the phase 2-form. For (1) we have (using equations (9) and (10) in [3])

$$\gamma_m(C) = -\iint_{S:\partial S=C} [V_{m,m+1} + V_{m,m-1}] \tag{9}$$

where

$$V_{m,n} = \frac{1}{(B)^2}\,\mathrm{Im}\,\frac{\langle m, j(\boldsymbol{B})|\,\mathrm{d}(\boldsymbol{J}\cdot\boldsymbol{B})|n, J(\boldsymbol{B})\rangle \wedge \langle n, J(\boldsymbol{B})|\,\mathrm{d}(\boldsymbol{J}\cdot\boldsymbol{B})|m, J(\boldsymbol{B})\rangle}{(m-n)^2}. \tag{10}$$

In order to evaluate the matrix elements in (10) we temporarily rotate axes so that the z-axis points along $\boldsymbol{B}$ and we get

$$V_{mn} = \frac{1}{2B^2}\{[J+m][J-m+1]\delta_{n,m+1} - [J-m][J+M+1]\delta_{n,m-1}\}\,\mathrm{D}B_x \wedge \mathrm{d}B_y. \tag{11}$$

Substituting (11) into (9) gives

$$\gamma_m(C) = -\tfrac{1}{2}[2m] \iint_{S:\partial S=C} \frac{\mathrm{d}B_x \wedge \mathrm{d}B_y}{B^2}. \tag{12}$$

Reverting to unrotated axes we obtain (in the familiar three-dimensional language)

$$\gamma_m(C) = -\tfrac{1}{2}[2m] \iint_S \frac{\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{B}}{B^2} \tag{13}$$

implying our promised result (8).

As another example consider the q-analogue of the displaced harmonic oscillator given by the Hamiltonian

$$\hat{H} = \hbar w[(a^+\alpha^*)(a+\alpha)+\tfrac{1}{2}]. \tag{14}$$

Here α and α^* are the adiabatic parameters, and a (a^+) is a q-analogue of the annihilation (creation) operator [1, 2]. This system can be considered as a contraction limit of the $SU(2)_q$ Hamiltonian

$$H = \hbar \boldsymbol{J}\cdot\boldsymbol{B} \tag{15}$$

with

$$J_\pm = \frac{J_x \pm \mathrm{i}J_y}{2} \qquad B_\pm = \frac{B_x \pm \mathrm{i}B_y}{2}. \tag{16}$$

From the matrix elements of H between $|jm\rangle$ states we have

$$H = \hbar B_z \sum_{m=-j}^{+j} m|jm\rangle\langle jm| + (\hbar B_-/\sqrt{2}\sqrt{[j-m][j+m+1]}\,|jm+1\rangle\langle jm|$$
$$+ \hbar B_+/\sqrt{2}\sqrt{[j+m][j-m+1]}\,|jm-1\rangle\langle jm|). \tag{17}$$

As long as the Hamiltonian matrix is finite dimensional, we are free to relabel the states

$$m \to m' = m+j \tag{18}$$

$$H = \hbar B_z \sum_{m'=0}^{2j} (m'-j)|jm'\rangle\langle m'j|$$
$$+ \hbar\sqrt{[j]} \sum_{m'=0}^{2j} \{[m'+1]^{1/2}|j, m'+1\rangle\langle j, m'| + [m']^{1/2}|j, m'-1\rangle\langle j, m'|\}$$
$$+ \mathrm{O}(\hbar/\sqrt{[j]}). \tag{19}$$

Now we take the contraction limit

$$j \to \infty$$

in such a way that

$$\sqrt{[j]}\, B_\pm/B$$

remain finite. We drop the term $\Sigma_{m'=0}^{2j} jB_z|jm'\rangle\langle jm'|$, an infinite renormalization of energy. In this limit the matrix elements become [1, 2]

$$m' = \langle m'|a^+a|m'\rangle$$
$$[m'+1] = \langle m'+1|a^+|m'\rangle$$
$$[m'] = \langle m'-1|a|m'\rangle$$

i.e. matrix elements of the q-analogues of the number, creation and destruction operators. Thus we reproduce the displaced oscillator (up to a constant which does not affect Berry's phase calculation):

$$H = \hbar w \sum_{m'} m'|j, m'\rangle\langle j, m'| + \sqrt{[m'+1]}\,\alpha|j, m'+1\rangle\langle j, m'|$$
$$+ \sqrt{[m']}\,\alpha^*|j, m'-1\rangle\langle jm'|)$$
$$= \hbar w(a^+a + \alpha a^+ + \alpha^* a) \tag{20}$$

with

$$w = B_z \approx B, \qquad \alpha = \lim_{j\to\infty} \frac{\sqrt{[j]}\, B-}{B} \tag{21}$$

appearing as the $j \to \infty$ limit of (15). In a similar spirit we can obtain from (11) the geometrical phase factor $\exp[\mathrm{i}\gamma_{m'}(C)]$ for the displaced harmonic oscillator

$$\gamma_m{}'(C) = -\iint_{S:\partial S=0} V_{m',m'+1} + V_{m',m'-1}$$
$$= [j]/B^2\{[m'+1]-[m']\}\,\mathrm{d}B_x \wedge \mathrm{d}B_y$$

which with $X = \sqrt{[j]}\, B_x/B$, $Y = -\sqrt{[j]}\, B_y/B$ becomes

$$\gamma_{m'}(C) = -2\{[m'+1]-[m']\} \iint_{S:\partial S=C} \mathrm{d}X \wedge \mathrm{d}Y \tag{22}$$

or finally after some algebraic manipulation we get

$$\gamma_{m'}(C) = -\frac{\cosh(2m'+1)s/2}{\cosh s/2} \oint_C (X\,\mathrm{d}Y - Y\,\mathrm{d}X). \tag{23}$$

In the limit $s \to 0$ (23) agrees with the result of Chaturvedi *et al* [5]. Note that quantum deformation of their system does not yield an m' independent result for Berry's phase. this is because the group theoretical multiplicative factors $[m']$ are not uniformlly spaced (for s different from zero).

Finally let us give the result of a Berry's phase calculation one expects to obtain for the generalized harmonic oscillator

$$H = \tfrac{1}{2}[Z(t)P^2 + Y(t)\{PQ+QP\} + X(t)Q^2] \tag{24}$$

where X, Y and Z are adiabatically varying parameters (with $XZ > Y^2$ and $Z > 0$), which is constructed from the q-momentum (P) and the q-position (Q) operators [1]

$$P = \mathrm{i}\sqrt{\hbar/2}\,(a^+ - a) \tag{25}$$

$$Q = \sqrt{\hbar/2}\,(a^+ + a). \tag{26}$$

Classically, this is an example of a particle racing around an elliptical curve in the phase space. Jackiw [6], de Sousa Gerbert [7] and Biswas [8] added the total derivative

$$\frac{\mathrm{d}f}{\mathrm{d}t} = -\frac{yQ^2}{2Z} \tag{27}$$

to the classical Lagrangian thereby inducing a canonical transformation to a new Hamiltonian

$$H' = \tfrac{1}{2}\{Z(P')^2 + (X - Y^2/Z - (\mathrm{d}/\mathrm{d}t)(Y/Z))Q'^2\} \tag{28}$$

i.e. without the term linear in momentum which is required to ensure the existence of a complex wavefunction and hence a non-zero Berry's phase. Through Berry's phase cannot arise from H', this is present in a new guise. The instantaneous energy levels of H are

$$E_n = \tfrac{1}{2}([n] + [n+1[)\,\hbar(ZX - Y^2)^{1/2} \tag{29}$$

while those of H' are

$$E'_n = \tfrac{1}{2}([n]+[n+1])\hbar\sqrt{Z\left(X-\frac{Y^2}{Z}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Y}{Z}\right)\right)}$$
$$= \tfrac{1}{2}([n]+[n+1])\hbar\sqrt{XZ-Y^2}\left[1-\frac{Z(\mathrm{d}/\mathrm{d}t)(Y/Z)}{2(ZX-Y^2)}+\ldots\right] \qquad (30)$$

where the dots refer to terms involving higher powers of the derivatives of the slow parameters. The difference between frequencies $E'_n/\hbar$ and $E_n/\hbar$, when integrated over one time period as part of the dynamical phase, without Berry's modification, yields the geometrical phase

$$\gamma_n(C) = -\left(\frac{[n]+[n+1]}{2}\right)\oint_C \frac{Y\,\mathrm{d}Z - Z\,\mathrm{d}Y}{2Z(ZX-Y^2)} \qquad (31)$$

for a closed circuit in Z, Y, X space. What we have accomplished at a semiclassical level by means of the canonical transformation (27) can also be extended to the quantum level via the corresponding unitary transformation $\exp(\mathrm{i}f/\hbar)$, where f is given by (27) (for a similar result see Giavarini *et al* [9]). In the limit $s \to 0$, (31) goes over to the result obtained by Berry [10].

Equations (8), (23) and (31) are the central results of this letter which are the q-analogues of corresponding results [3, 5, 10] for prototype calculations of the Berry phase. Since, loosely speaking, the deformation parameter $q = \exp(\hbar)$ we expect that the semiclassical limit of the geometrical phase is not altered upon quantum deformation of the classical system.

It is a pleasure to acknowledge the Council of Scientific and Industrial Research for financial support.

References

[1] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[2] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[3] Berry M V 1984 *Proc. R. Soc.* A **392** 45
[4] Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 *Lett. Math. Phys.* **11** 247
[5] Chaturvedi S, Sriram M S and Srinivasan V 1987 *J. Phys. A: Math. Gen.* **20** L1071
[6] Jackiw R 1988 *Int. J. Mod. Phys.* A **3** 285
[7] de Sousa Gerbert Ph 1989 *Ann. Phys., NY* **189** 155
[8] Biswas S N 1989 lecture presented at Vishwabharti, Santiniketan (to be published)
[9] Giavarini G, Gozzi E, Rohrlich D and Thacker W D 1989 *J. Phys. A: Math. Gen.* **22** 3513
[10] Berry M V 1985 *J. Phys. A: Math. Gen.* **18** 15